Received: *November 10, 1999*, Accepted: *November 11, 1999*

On six dimensional "gauge" theories with irrational theta angle

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Abstract: A recent proposal for 6d "gauge" theories with rational theta angle is discussed. These theories were constructed from n_C coinciding (p, q) 5-branes of IIB in the limit of vanishing string constant. They have $(1, 1)$ supersymmetry and the low energy theory is 6d Yang-Mills with gauge group $\mathrm{SU}(n_C)$ and a rational theta angle $\theta_6/2\pi = f/q$, where $fp = 1 \pmod q$. By changing the point of view, considering the (p, q) 5-branes to be D5-branes, the 6d theta angle is identified with a 10d theta angle. This point of view together with some assumptions suggests a generalization of the previous limit to arbitrary theta. This limit seems to define a decoupled 6d theory even though the 10d theory does not become free in general.

Keywords: Field Theories in Higher Dimensions, Brane Dynamics in Gauge Theories.

JHEP11(1999)017

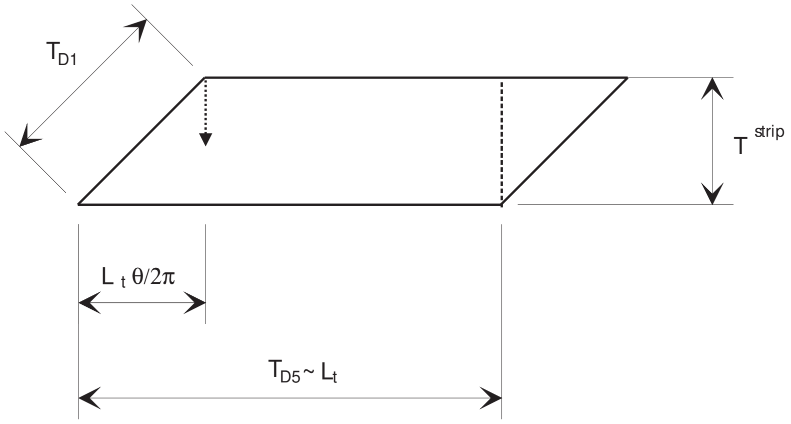

Figure 1: The small torus of IIB with base L_t and a theta angle. The shown dimensions are proportional to the D5-brane tension T_{D5}, the strip tension T^{strip} (dashed), and the D-string tension T_{D1}. The limit $\mathrm{Im}(\tau) \to 0$ is demonstrated by the dotted arrow.

Six dimensional Theories without gravity were introduced in [1, 2]. Related ideas were discussed in studies of black holes, BPS charges and strings inside 5-branes [3, 4, 5]. These theories were found to play a role in matrix M theory [6, 7]. Consider those with (1,1) supersymmetry. Such theories reduce in low energies to 6d Yang-Mills, and their lagrangean includes the following terms

$$\mathcal{L} = \frac{1}{g_6^2} F^2 + \frac{\theta_6}{2\pi} F \wedge F \wedge F + \cdots , \tag{1}$$

where F is the field strength, g_6 is the 6d coupling constant, and θ_6 is a 6d theta angle, related to the π_5 homotopy of the gauge group. The energy scale in the 5-brane is set by $1/g_6^2$, the tension of the instanton inside the 5-brane. Some constructions of these theories in terms of the 5-branes of type-IIB [8, 9] will be discussed here.

Let us set the notations to discuss type-IIB. Denote the complex scalar of IIB by $\tau = \theta_B/2\pi + i/\lambda$, where λ is the string coupling and θ_B is the IIB theta angle. Type-IIB on a large circle of size L_B with a complex scalar τ is the limit of M theory compactified on a small torus with base L_t and modular parameter τ [10, 11] (figure 1). Let A_r denote the torus area $A_r = L_t^2\,\mathrm{Im}(\tau)$. The compactification sizes are related through $2\pi/L_B = A_r T_M = L_t^2\,\mathrm{Im}(\tau) M_{11}^3/(2\pi)^2$, where T_M is the M-membrane tension,[1] and M_{11} is the 11d Planck length. The 10d Planck length M_p is given by $L_B M_p^8 = A_r M_{11}^9$. An expression that depends on the M theory variables survives the IIB limit if when translated to IIB variables it is independent of L_B. A (p, q) 5-brane[2] can have an instantonic string inside it. This instanton was called a "strip" in [12] since its M theory origin is a membrane stretched over an interval with

[1] The conventions used for 2π factors are: $T_M = M_{11}^3/(2\pi)^2$, $T_{M5} = M_{11}^6/(2\pi)^5, T_{Dp} = 1/[\lambda(2\pi)^p \alpha'^{(p+1)/2}]$.

[2] Wrapping M-branes around (p, q) cycles produces branes of IIB. A $(1, 0)$ string is the fundamental string with tension $T_s = L_t T_M$, whereas a $(0, 1)$ string is the D-string. The tension of

endpoints on the M5-brane (figure 1). The strip tension in a general (p,q) 5-brane with a general τ is given by

$$T^{strip}_{p,q} = \frac{A_r}{|p+\tau q|\, L_t} T_M = \frac{\mathrm{Im}(\tau)}{|p+\tau q|} T_s\,. \tag{2}$$

In [8] Seiberg introduced a number of new 6d constructions. The relevant one to us is constructed from

$$\text{A:} \begin{cases} n_C \text{ coinciding NS5-branes,} \\ \text{taking the limit } \lambda \to 0\,. \end{cases}$$

The low energy gauge group is $\mathrm{SU}(n_C)$. For the NS5-brane the tension of the instanton inside the 5-brane, the only energy scale, is (2) $T^{strip}_{NS5} = 1/g_6^2 = T_s$. Since this tension is independent of λ, it was argued that for $\lambda \to 0$, the bulk becomes free and a non-trivial 6d theory decouples from it.

To describe this theory in the IIB limit of M theory, we should perform the limit $\lambda \to 0$ while *fixing the compactification size in 6d units* $L_B\sqrt{T^{strip}} = \epsilon^{-1}$. For a theory on an NS5-brane the limit translates to $L_t \to 0$, $\tau = i\epsilon/L^{3/2}(2\pi/T_M^{3/2})$. After taking this limit we can take the decompactification limit $\epsilon \to 0$.

The appearance of *fractional instantons* of tension $T \sim T^{strip}/n_C$ [13] is natural in this picture.[3] In 11d the circle transverse to the M5-branes is small and the M5-branes could be distributed around it in various ways, while still coinciding in the 10d sense. A general configuration, roughly equally spaced around the circle, results in n_C fractional strips. These correspond to membranes ending on adjacent M5-branes. The existence of fractional strips can be ignored in the following discussion since n_C will be fixed.

The discussion was generalized by Witten [9] to describe theories parametrized by integers f, q, n_C where f and q are relatively prime and $0 \le f \le q-1$. The low energy description of these theories has $\mathrm{SU}(n_C)$ gauge group, and a rational theta angle $\theta_6/2\pi = f/q$. Define integers p, e such that

$$M = \begin{bmatrix} p & e \\ q & f \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z}). \tag{3}$$

Explicitly, p is given by $pf = 1 \pmod q$, and we have the freedom $(p,e) \to (p,e) + n(q,f), n \in \mathbb{Z}$. (Note: the notation used here is slightly different than in [9]). These theories are constructed in type IIB as the theory of

$$\text{B:} \begin{cases} n_C \text{ coinciding } (p,q) \text{ 5-branes,} \\ \text{taking the limit } \tau \to i\infty\,, \\ \text{or } L_t \to 0,\ \tau \sim \frac{i\epsilon}{L^{3/2}\sqrt{q}} \cdot \frac{2\pi}{T_M^{3/2}}\,. \end{cases}$$

a general (p,q) string is given by $T^{string}_{p,q} = |p+q\tau|\, L_t T_M$. A general (p,q) 5-brane has tension $T^{5-brane}_{p,q} = |p+q\tau|\, \mathrm{Im}(\tau) T_s^3/(2\pi)^2 = |p+q\tau|\, \frac{L_t}{L_B} M_{11}^6/(2\pi)^5$, where $(1,0)$ is the D5-brane and $(0,1)$ is the NS5-brane.

[3] The author thanks I. Klebanov, J. Maldacena and A. Rajaraman for discussing this point.

As in the previous case the bulk theory becomes free, and the 6d is decoupled and non-trivial.

Let us change the point of view and consider the n_C 5-branes to be n_C D5-branes, through an $\mathrm{SL}(2,\mathbb{Z})$ transformation. Explicitly, the required transformation is $M^{-1}(3)$. Then the construction consists of

$$\text{C:} \quad \begin{cases} n_C \text{ coinciding D5-branes,} \\ \text{taking the limit } \tau \to 0 \text{ on the imaginary axis, for the first case A,} \\ \text{and } \tau \to \frac{f}{q} \text{ with fixed real part, for the second case B,} \\ \text{or } L_t \to \infty, \tau = Re(\tau) + i\frac{\epsilon^2}{L_t^3} \cdot \frac{(2\pi)^2}{T_M} . \end{cases}$$

The advantage in this point of view is the direct relation

$$\theta_B = \theta_6 \,, \tag{4}$$

which can be read off the D5-brane action. Note that in this limit $\lambda \to \infty$ which is surprising at first.

Let us analyze the conditions for the 10d theory to be free and for the 6d theory to decouple. *The 10d theory is free* if there is some (l,k) string with a small string coupling, so its tension satisfies

$$T^{string}_{l,k} \ll M_p^2 \,. \tag{5}$$

In the limit $\tau \to f/q$ it is the $(f,-q)$ string that becomes light. Its tension, as $L_t \to \infty$ is

$$T^{string}_{f,-q} \sim q\,\mathrm{Im}(\tau) L_t T_M = \frac{q\epsilon^2}{L_t^2}\frac{(2\pi)^2}{T_M} \,. \tag{6}$$

The tension of the strip, T^{strip} and the tension of the 5-brane, T_5, are related through

$$T^{strip} \cdot T_5 = M_p^8 \,. \tag{7}$$

This relation is a result of Dirac quantization in 11d between the M-membrane and its magnetic dual, the M5-brane. So the energy scales in the 6d theory are characterized by a single scale, and it seems that *the condition for the 6d theory to decouple* is

$$T^{strip} = 1/g_6^2 = \frac{A_r}{|p+\tau q|\, L_t} T_M \ll M_p^2 \,. \tag{8}$$

From this point of view *a generalization to irrational theta angles suggests itself.* Consider taking

$$\text{D:} \quad \begin{cases} n_C \text{ coinciding D5-branes,} \\ \text{and the limit } \tau \to \frac{\theta_6}{2\pi} \\ \text{or } L_t \to \infty, \tau = \frac{\theta_6}{2\pi} + i\frac{\epsilon^2}{L_t^3} \cdot \frac{(2\pi)^2}{T_M} . \end{cases}$$

θ_6 is arbitrary now and p, q are not required. In order to consider the conditions (5) and (8) we should measure distances on the torus plane divided by M_p^2. In these units the limit turns out to be $L_t \to \infty$, $\mathrm{Im}(\tau) = 1/L_t$ (suppressing M_{11}), so that the area of the torus is fixed in these units. We see that the condition for the 6d theory to decouple (8) is satisfied. Assuming the mentioned procedures for defining 6d theories are correct, and that (8) is indeed the correct condition for a 6d theory to decouple, we have a theory with an arbitrary theta angle. This became possible by considering tori that do not necessarily generate a square lattice.

This theory does not allow to define (in general) a limit in which the bulk is free. It turns out that taking the limit for τ while measuring tensions in Planck units corresponds to the torus degenerating while keeping its area fixed. To have a free limit we should look for light strings. Consider an (l, k) string. Its tension is $\left|(l + k\frac{\theta_6}{2\pi})L_t + ik/L_t\right|$ in 10d Planck units. As a function of L_t it has a minimum value of

$$T_{l,k} = \sqrt{k\left(l + k\frac{\theta_6}{2\pi}\right)}\sqrt{2}\,. \tag{9}$$

An irrational number, x, is said to be *approximated by rationals to order* n [14] if $\exists c > 0$, and infinitely many integers k such that

$$|kx(mod 1)| < \frac{c}{k^{(n-1)}}\,. \tag{10}$$

We see that only if the degree of rational approximation for $\theta_6/2\pi$ is greater than two, $n > 2$, a limit can be defined such that the condition for the bulk theory to be free (5) is satisfied. It turns out that for a general irrational $n = 2$ with a constant $c = 1/\sqrt{5}$, and so in general a free limit cannot be defined. Special irrationals with higher n should allow a free bulk. Does an M theory construction [9] exist for them?

Let us *discuss the role of rationals* in these theories. It seems that there is *a continuous transition between theories of different (rational)* $\theta_6/2\pi$ thus answering a question raised in [9]. From a physical point of view note that the transition from rationals to irrationals is smooth. From (6) we see that as the rational denominator, q, grows, the free bulk limit is reached later (for higher L_t). Finally, when the number is irrational this limit cannot be reached anymore. It is interesting to note another "rational" physical phenomena - only for rational $\theta_6/2\pi$ it is possible for a numbers of strips to join, become a string and leave the 5-brane.

Acknowledgments

The author thanks O. Aharony, E. Halyo, A. Rajaraman, L. Susskind and E. Witten. B.K. is supported by NSF grant PHY-9219345.

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